A Fixed-Gaussian Two-Point Criterion for the Riemann Hypothesis

A Classical Explicit-Formula Synthesis and Two-Translate Test

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Abstract

For each fixed $a > 0$ we form an even real-analytic function K_a by applying a Gaussian weight to the nontrivial zeros of the Riemann zeta function and translating the resulting cosine sum. We give a direct proof that the Riemann hypothesis is equivalent to any of three statements: boundedness of K_a on the nonnegative real axis, polynomial growth of K_a, or positive semidefiniteness of a 2×2 matrix constructed from $K_a(0)$ and $K_a(t)$ for every real t. The converse uses the meromorphic continuation of the Laplace transform: any off-critical zero produces a pole in the open right half-plane that is incompatible with polynomial growth. The Guinand–Weil explicit formula yields an unconditional expression for K_a involving a hyperbolic pole term, a digamma integral, and a Gaussian-smoothed sum over prime powers. We describe a validated pointwise numerical procedure, distinguish this criterion from general two-point tests for positive definiteness, and explain its limitations. The argument synthesizes classical explicit-formula and growth criteria with a two-translate restriction of Weil positivity. In particular, it is compared with Suzuki's unregularized boundedness and pointwise-sign criteria. A finite continuous-interval two-point positivity result is also derived from previously verified low zeros, explicit zero-counting bounds, and a reproducible two-ordinate interval certificate. No proof of the Riemann hypothesis, claim of priority, or new prime-number estimate is made.

1 Introduction and scope

Let $\zeta(s)$ denote the Riemann zeta function and let its nontrivial zeros be counted with multiplicity. The Riemann hypothesis (RH) asserts that each such zero has real part $\frac{1}{2}$. Weil's positivity criterion and the Guinand–Weil explicit formula are classical ways of relating this problem to harmonic analysis and the distribution of prime powers [2, 3, 4, 6, 5]. Positivity and transform criteria have also been studied via the screw function attached to the Riemann zeta function [7].

The purpose of this note is to isolate a small *detector* of off-critical zeros: two-point matrices generated by one fixed Gaussian. Boundedness criteria obtained from the explicit formula are classical, and Suzuki has proved particularly strong analogous criteria for an unregularized screw function; see Section 6. The reduction is lossless in the precise sense that every violation of RH produces a failure at some separation t. It does *not* supply the missing uniform estimate needed to prove RH. While restricting the test family to two translates of a single Gaussian yields a particularly simple 2×2 matrix criterion, the underlying boundedness and positivity mechanisms belong to a well-established explicit-formula tradition [4, 7].

Our convention for the Fourier transform is

$$\widehat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{iux}\, dx, \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(u) e^{-iux}\, du. \tag{1}$$

2 The Gaussian zero kernel

For every nontrivial zero ρ put

$$\lambda_\rho = \rho - \tfrac{1}{2}, \qquad z_\rho = \frac{\rho - \frac{1}{2}}{i} = -i\lambda_\rho. \tag{2}$$

The multiset of λ_ρ is invariant under $\lambda \mapsto -\lambda$ and under complex conjugation. For a fixed $a > 0$, define

$$K_a(t) = \sum_\rho e^{-a^2 z_\rho^2} \cos(t z_\rho) = \sum_\rho e^{a^2 \lambda_\rho^2} \cosh(t\lambda_\rho), \qquad t \in \mathbb{R}. \tag{3}$$

The sums always run over nontrivial zeros, with multiplicity. We do not assume RH in the definition.

Lemma 2.1 (Regularity and symmetry)**.** *For fixed $a > 0$, the series in (3) converges locally uniformly for complex t. It defines an entire even function of t, real-valued on the real axis. Moreover,*

$$K_a(t) = \sum_\rho e^{a^2 \lambda_\rho^2} e^{t\lambda_\rho}, \qquad t \in \mathbb{C}. \tag{4}$$

For real t there is a constant $C_a < \infty$ such that $|K_a(t)| \le C_a e^{|t|/2}$.

Proof. Write $\rho = \beta + i\gamma$. Since $0 < \beta < 1$ for nontrivial zeros, $|\operatorname{Re}\lambda_\rho| < \frac{1}{2}$. Also

$$\left|e^{a^2\lambda_\rho^2}\right| = e^{a^2((\beta-1/2)^2 - \gamma^2)} \le e^{a^2/4} e^{-a^2\gamma^2}.$$

For t in a compact subset of $\mathbb{C}$, both $\cosh(t\lambda_\rho)$ and $e^{t\lambda_\rho}$ are bounded by a constant times $e^{C|\gamma|}$, with C depending on the compact set. The standard zero counting bound $N(T) = O(T\log(T+2))$ proves locally uniform absolute convergence. The symmetries of the zero multiset imply evenness and reality on $\mathbb{R}$. Pairing λ with $-\lambda$ proves (4). The last assertion follows from the same estimate with $t \in \mathbb{R}$. □

3 The lossless two-point criterion

Define the symmetric real matrix

$$G_a(t) = \begin{pmatrix} K_a(0) & K_a(t) \\ K_a(t) & K_a(0) \end{pmatrix}. \tag{5}$$

Theorem 3.1 (A fixed-Gaussian criterion)**.** *Fix* any one $a > 0$. *The following are equivalent:*

(i) *The Riemann hypothesis is true.*

(ii) *$G_a(t)$ is positive semidefinite for every $t \in \mathbb{R}$.*

(iii) *$|K_a(t)| \le K_a(0)$ for every $t \in \mathbb{R}$.*

(iv) *K_a is bounded on $[0, \infty)$.*

(v) *For some finite $A \ge 0$, one has $K_a(t) = O((1+t)^A)$ as $t \to +\infty$.*

In particular, if RH *is false, $\sup_{t\ge 0} |K_a(t)| = \infty$, and K_a violates every polynomial growth bound.*

Proof. Assume RH. Then every z_ρ is real and every Gaussian coefficient $e^{-a^2 z_\rho^2}$ is strictly positive. By the triangle inequality,

$$|K_a(t)| \le \sum_\rho e^{-a^2 z_\rho^2} = K_a(0) < \infty.$$

The eigenvalues of (5) are $K_a(0) + K_a(t)$ and $K_a(0) - K_a(t)$; hence (i) implies (ii) and (iii), while (ii) and (iii) are equivalent. The implications (iii)⇒(iv)⇒(v) are immediate.

For the converse suppose (v). By Lemma 2.1, K_a has the exponential representation (4). For $\operatorname{Re} s > \frac{1}{2}$ we may integrate termwise and obtain

$$\mathcal{L}K_a(s) = \int_0^\infty e^{-st} K_a(t)\, dt = \sum_\rho \frac{e^{a^2\lambda_\rho^2}}{s - \lambda_\rho}. \tag{6}$$

The right-hand side defines a meromorphic function of $s \in \mathbb{C}$: the zero-counting estimate and Gaussian decay imply locally normal convergence away from the discrete set of poles $s = \lambda_\rho$. If a zero ρ_0 has $\operatorname{Re}\rho_0 > \frac{1}{2}$, then $s_0 = \lambda_{\rho_0}$ lies in $\operatorname{Re} s > 0$ and the pole at s_0 has nonzero residue

$$m(\rho_0) e^{a^2\lambda_{\rho_0}^2},$$

where $m(\rho_0) \ge 1$ is its multiplicity. No cancellation is possible: all zeros producing the same pole have the same nonzero Gaussian coefficient.

On the other hand, polynomial growth of K_a implies that the integral on the left of (6) converges locally uniformly and defines a holomorphic function throughout $\operatorname{Re} s > 0$. By uniqueness of analytic continuation, this contradicts the pole at s_0. Thus there is no zero to the right of the critical line. The functional equation gives symmetry of zeros under $\rho \mapsto 1 - \rho$, so none lies to the left either. Hence RH holds. □

Remark 3.2 (Why the cosine transform matters). Directly transforming an individual $\cos(tz_\rho)$ produces a combination of resolvents. The single-resolvent expression in (6) is valid because the *full* zero multiset is invariant under $\lambda \mapsto -\lambda$, and therefore (4) is an exact identity. This pairing must not be omitted when deriving the pole argument.

Remark 3.3 (No assumption of a maximal off-line real part). The proof does not require the supremum of the real parts of off-line zeros to be attained. A single zero with real part greater than $\frac{1}{2}$ suffices to create a nonremovable pole. This avoids making unsupported assertions about a hypothetical outermost zero.

4 The unconditional prime-side formula

Let $\Lambda(n)$ be the von Mangoldt function, and let

$$\Omega(u) = \operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log\pi.$$

We use the standard Guinand–Weil explicit formula [2, 3, 4, 6, 5]: for an admissible even transform pair $f, \widehat{f}$,

$$\sum_\rho \widehat{f}(z_\rho) = \widehat{f}(i/2) + \widehat{f}(-i/2) - \sum_{n=2}^\infty \frac{\Lambda(n)}{\sqrt{n}}\left(f(\log n) + f(-\log n)\right) + \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(u)\Omega(u)\, du. \tag{7}$$

The Gaussian functions used below satisfy the necessary decay hypotheses, including the required decay in the vertical strips in which zeros lie.

Proposition 4.1 (Arithmetic representation)**.** *For every fixed $a > 0$ and every real t,*

$$K_a(t) = 2e^{a^2/4}\cosh(t/2) + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a^2u^2}\cos(tu)\Omega(u)\,du \tag{8}$$
$$- \frac{1}{2a\sqrt{\pi}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left[e^{-(t-\log n)^2/(4a^2)} + e^{-(t+\log n)^2/(4a^2)}\right].$$

Both the prime-power series and the archimedean integral are absolutely convergent for each fixed t.

Proof. Take

$$\widehat{f}_{a,t}(u) = e^{-a^2u^2}\cos(tu).$$

Under (1), its inverse Fourier transform is the even Gaussian pair

$$f_{a,t}(x) = \frac{1}{4a\sqrt{\pi}}\left[e^{-(x-t)^2/(4a^2)} + e^{-(x+t)^2/(4a^2)}\right]. \tag{9}$$

The zero side of (7) is $K_a(t)$. The pole terms are $\widehat{f}_{a,t}(\pm i/2) = e^{a^2/4}\cosh(t/2)$, giving the first term of (8). Since $f_{a,t}$ is even, the prime contribution is $-2\sum_{n\geq 2}\Lambda(n)f_{a,t}(\log n)/\sqrt{n}$, giving the stated coefficient $1/(2a\sqrt{\pi})$. The remaining term is the displayed archimedean integral. Gaussian decay gives convergence and justifies the substitution. □

Corollary 4.2 (An exact arithmetic inequality equivalent to RH)**.** *For any fixed $a > 0$,* RH *holds if and only if the real expression on the right of* (8) *satisfies*

$$|K_a(t)| \leq K_a(0) \quad \textit{for all } t \geq 0,$$

where $K_a(0)$ is computed by the same arithmetic expression at $t = 0$.

5 Positivity, spectral reality, and what is not proved

For an arbitrary real even continuous function K, the condition $|K(t)| \leq K(0)$ is not equivalent to positive definiteness: Bochner's theorem in general requires nonnegative matrices $[K(t_i - t_j)]_{i,j=1}^n$ for every n and all choices of sample points. Here two-point testing suffices *only* because of the special zero expansion (3): on RH the associated spectral weights are positive, whereas off-line zeros generate poles in the right half-plane of the Laplace transform.

Proposition 5.1 (Full positive definiteness is equivalent too)**.** *For the particular family K_a in* (3)*,* RH *is equivalent to positive definiteness of K_a in the sense of Bochner.*

Proof. Under RH, the symmetrized atomic measure $\sum_\rho e^{-a^2z_\rho^2}\delta_{z_\rho}$ is positive, symmetric, and finite, and K_a is its Fourier transform. Conversely positive definiteness implies the 2×2 condition of Theorem 3.1. □

Proposition 5.2 (Exact partial-summation identity)**.** *For $a > 0$, $t \geq 0$, and $u \geq 1$, define the logarithmic Gaussian weight*

$$W_{a,t}(u) = \frac{u^{-1/2}}{2a\sqrt{\pi}}\left[e^{-(t-\log u)^2/(4a^2)} + e^{-(t+\log u)^2/(4a^2)}\right]$$

and the prime-counting error $E(u) = \psi(u) - (u-1)$*, where* $\psi(u) = \sum_{n \le u} \Lambda(n)$*. Then the prime-power sum* $S_a(t)$ *in* (8) *satisfies the exact identity*

$$S_a(t) = \int_1^\infty W_{a,t}(u)\,du - \int_1^\infty E(u) W'_{a,t}(u)\,du, \tag{10}$$

where the continuous main term evaluates in closed form to

$$\int_1^\infty W_{a,t}(u)\,du = \frac{e^{a^2/4}}{2}\left[e^{t/2}\operatorname{erfc}\left(-\frac{t+a^2}{2a}\right) + e^{-t/2}\operatorname{erfc}\left(\frac{t-a^2}{2a}\right)\right]. \tag{11}$$

Consequently, $P_a(t) - \int_1^\infty W_{a,t}(u)\,du = O_a(e^{-t/2})$ *as* $t \to +\infty$*, and for every fixed* $B > 0$*,*

$$K_a(t) = \int_1^\infty E(u) W'_{a,t}(u)\,du + O_{a,B}((1+t)^{-B}) \qquad (t \ge 0). \tag{12}$$

Proof. Writing $S_a(t) = \int_{1^+}^\infty W_{a,t}(u)\,d\psi(u)$ and substituting $\psi(u) = (u-1) + E(u)$, Stieltjes integration by parts gives (10). Indeed, $E(1) = 0$, and the elementary bound $\psi(u) = O(u\log(2u))$ suffices to show that $E(u)W_{a,t}(u) \to 0$ as $u \to \infty$. Substituting $x = \log u$ in $\int_1^\infty W_{a,t}(u)\,du$ and completing the square proves (11). Using $\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$ gives

$$P_a(t) - \int_1^\infty W_{a,t}(u)\,du = \frac{e^{a^2/4}}{2}\left[e^{t/2}\operatorname{erfc}\left(\frac{t+a^2}{2a}\right) + e^{-t/2}\left(2 - \operatorname{erfc}\left(\frac{t-a^2}{2a}\right)\right)\right] = O_a(e^{-t/2}).$$

Finally, $u \mapsto e^{-a^2u^2}\Omega(u)$ is smooth with Gaussian decay of all derivatives on $\mathbb{R}$. Repeated integration by parts in the archimedean integral therefore gives $H_a(t) = O_{a,B}((1+t)^{-B})$ for every $B > 0$. Combining $K_a = P_a + H_a - S_a$ with (10) proves (12). □

Remark 5.3 (Interpretation of the arithmetic cancellation). Proposition 5.2 separates two distinct phenomena in (8). First, the exponentially growing pole term $P_a(t) \sim e^{a^2/4}e^{t/2}$ is canceled to within $O_a(e^{-t/2})$ by the continuous main term $u - 1$ of $\psi(u)$. Second, by Theorem 3.1(v) and (12), RH is equivalent to the assertion that the Gaussian-smoothed prime-error integral $\int_1^\infty E(u)W'_{a,t}(u)\,du$ grows at most polynomially in t (or remains bounded) as $t \to +\infty$. Because $W'_{a,t}(u)$ is localized near $\log u \approx t$, with a factor of order $u^{-3/2}$ times a Gaussian and a polynomial in $t - \log u$, this reflects the classical connection between RH and the square-root error bound $E(u) = O(u^{1/2}\log^2 u)$, smoothed at scale $u \approx e^t$. The prime sum is infinite despite this Gaussian localization; numerical truncations still require rigorous tails.

Proposition 5.4 (Logarithmic convolution and the unconditional baseline)**.** *Fix* $a > 0$*, and set*

$$\begin{aligned}
\widetilde{E}(x) &= e^{-x/2}E(e^x) \qquad (x \ge 0),\\
\phi_a(y) &= \frac{1}{2a\sqrt{\pi}}e^{-y^2/(4a^2)},\\
\kappa_a(y) &= -\tfrac{1}{2}\phi_a(y) - \phi_a'(y) = \frac{y-a^2}{4a^3\sqrt{\pi}}e^{-y^2/(4a^2)}.
\end{aligned}$$

Then the operator $\mathcal{T}_a[E](t) = \int_1^\infty E(u)W'_{a,t}(u)\,du$ *has the exact representation*

$$\mathcal{T}_a[E](t) = \int_0^\infty \widetilde{E}(x)\kappa_a(t-x)\,dx + \int_0^\infty \widetilde{E}(x)\kappa_a(-t-x)\,dx. \tag{13}$$

For $t \to +\infty$, the reflected term is $O_a(e^{-t^2/(4a^2)})$. Furthermore, there is a constant $c > 0$ such that, unconditionally,

$$\mathcal{T}_a[E](t),\ K_a(t) \ll_a \exp\left(\frac{t}{2} - c\frac{t^{3/5}}{(\log t)^{1/5}}\right) \qquad (t \geq 3). \tag{14}$$

The kernel does not have mean zero; indeed

$$\int_{\mathbb{R}} \kappa_a(y)\,dy = -\frac{1}{2}, \qquad \int_{\mathbb{R}} \kappa_a(y)e^{-\lambda y}\,dy = -(\lambda + \tfrac{1}{2})e^{a^2\lambda^2} \quad (\lambda \in \mathbb{C}). \tag{15}$$

Proof. Since $W_{a,t}(e^x) = e^{-x/2}[\phi_a(t-x) + \phi_a(t+x)]$, the chain rule gives

$$e^x W'_{a,t}(e^x) = e^{-x/2}\big[\kappa_a(t-x) + \kappa_a(-t-x)\big].$$

Multiplication by $E(e^x) = e^{x/2}\widetilde{E}(x)$ and substitution $u = e^x$ prove (13). The elementary estimate $\psi(u) \ll u\log(2u)$ gives $|\widetilde{E}(x)| \ll e^{x/2}(1+x)$. Hence the Gaussian bound on $\kappa_a(-t-x)$, with $x \geq 0$, gives $O_a(e^{-t^2/(4a^2)})$ for the reflected integral as $t \to +\infty$.

The classical Korobov–Vinogradov prime-number-theorem error bound [5] implies, for some $c_0 > 0$,

$$|\widetilde{E}(x)| \ll \exp\left(\frac{x}{2} - c_0\frac{x^{3/5}}{(\log x)^{1/5}}\right) \qquad (x \geq 3).$$

For t sufficiently large, split the first integral in (13) into $t/2 \leq x \leq 3t/2$ and its complement. On the central interval the function $x^{3/5}/(\log x)^{1/5}$ is bounded below by a fixed positive multiple of its value at t, and the factor $e^{(x-t)/2}|\kappa_a(t-x)|$ has a uniformly bounded integral. On the complementary intervals the elementary bound on $\widetilde{E}$ and Gaussian tails give a stronger $O_a(e^{-c_a t^2})$ estimate. Shrinking c_0 if necessary proves (14) for $\mathcal{T}_a[E]$. Equation (12) then gives the same estimate for K_a. Finally, integration of ϕ_a and of its derivative, followed by integration by parts against $e^{-\lambda y}$, gives (15). □

Remark 5.5 (Fixed-width smoothing and growth exponents). The Gaussian in (13) is localized on a *fixed* logarithmic scale $|x-t| = O(a)$; it does not widen to $O(a\sqrt{t})$. In particular, (15) shows that the kernel is not a zero-mean wavelet and does not by itself annihilate slowly varying errors. On an exponential mode $e^{\lambda x}$, its full-line convolution acts by the multiplier $-(\lambda + \frac{1}{2})e^{a^2\lambda^2}$. For a nontrivial zeta zero, $\lambda = \rho - \frac{1}{2}$, this multiplier is $-\rho e^{a^2\lambda^2} \neq 0$. Thus smoothing suppresses large imaginary parts, but does not remove the exponential growth rate of an individual off-line zero.

For any $0 < \delta < 1/2$, the growth estimate

$$\mathcal{T}_a[E](t) \ll_{a,\delta} e^{(1/2-\delta)t} \qquad (t \to +\infty)$$

for one fixed $a > 0$ is equivalent to the zero-free strip $\operatorname{Re}\rho \leq 1 - \delta$ for all nontrivial zeros: the forward implication uses Theorem 3.1's Laplace-pole argument and (12); the reverse implication follows by absolute convergence of the Gaussian zero trace. By contrast, a merely subexponential saving $e^{t/2-ct^\alpha}$ with $0 < \alpha < 1$ does *not* by itself extend the ordinary Laplace integral into a fixed half-plane $\operatorname{Re} s < 1/2$. Stronger subexponential error estimates could still be significant, but relating them to a quantitative height-dependent zero-free region requires additional arguments, rather than just this Laplace-pole test.

Proposition 5.6 (Verified-zero splitting and a stronger Gaussian baseline)**.** *Let $a > 0$, and suppose all zeros $\rho = \beta + i\gamma$ with $|\gamma| \leq T_0$ are known to satisfy $\beta = \frac{1}{2}$. For $0 < \theta < 1$, put*

$$Z_{a,\theta} = \sum_\rho e^{-\theta a^2\gamma^2} < \infty, \quad B_{a,T_0} = \sum_{|\gamma|\leq T_0} e^{-a^2\gamma^2}, \quad K_a^{\leq T_0}(t) = \sum_{|\gamma|\leq T_0} e^{a^2\lambda_\rho^2 + t\lambda_\rho}.$$

Then $|K_a^{\le T_0}(t)| \le B_{a,T_0}$ *for real* t*, and for every* $t \ge 0$*,*

$$\left|K_a(t) - K_a^{\le T_0}(t)\right| \le e^{a^2/4} Z_{a,\theta} \exp\left(\frac{t}{2} - (1-\theta)a^2 T_0^2\right). \tag{16}$$

In particular, when $0 \le t \le a^2 T_0^2$*,*

$$\left|K_a(t) - K_a^{\le T_0}(t)\right| \le e^{a^2/4} Z_{a,1/4} e^{-a^2 T_0^2/4}. \tag{17}$$

Let $R = 57.54$ *(Ford's explicit constant [10]; see also Bellotti [12] for the refinement* $R = 53.989$*) and define, for* $T \ge T_0$*,*

$$\eta_R(T) = \frac{1}{R(\log T)^{2/3}(\log\log T)^{1/3}}.$$

Assume $T_0 \ge 3$*. For any* $T \ge T_0$ *and* $t \ge 0$*,*

$$|K_a(t) - K_a^{\le T_0}(t)| \le e^{a^2/4} Z_{a,1/2} \Big[\mathbf{1}_{\{T > T_0\}} \exp\Big(\Big(\tfrac{1}{2} - \eta_R(T)\Big) t - \tfrac{1}{2} a^2 T_0^2\Big) + \exp\Big(\tfrac{t}{2} - \tfrac{1}{2} a^2 T^2\Big) \Big]. \tag{18}$$

Consequently, for every fixed $a > 0$ *and any fixed* $0 < c < 2^{2/3}/57.54$*, as* $t \to \infty$*,*

$$|K_a(t)|,\ |\mathcal{T}_a[E](t)| \ll_{a,c} \exp\left(\frac{t}{2} - \frac{ct}{(\log t)^{2/3}(\log\log t)^{1/3}}\right). \tag{19}$$

The implied constant and the threshold for this final estimate may depend on a *and* c*.*

Proof. By the absolute zero trace (4),

$$|e^{a^2\lambda_\rho^2 + t\lambda_\rho}| \le e^{a^2/4 - a^2\gamma^2 + t(\beta - 1/2)}.$$

The verified zeros have $\operatorname{Re}\lambda_\rho = 0$, giving the asserted bound for $K_a^{\le T_0}$. For $|\gamma| > T_0$, use $\beta - 1/2 < 1/2$ and

$$e^{-a^2\gamma^2} \le e^{-(1-\theta)a^2 T_0^2} e^{-\theta a^2 \gamma^2}.$$

Summing proves (16) and, with $\theta = 1/4$, (17).

For the three-zone bound, separate the unknown zeros into $T_0 < |\gamma| \le T$ and $|\gamma| > T$. Ford's zero-free region [10] implies $\beta \le 1 - \eta_R(T)$ on the intermediate range; on the final range use $\beta < 1$. For either range use $e^{-a^2\gamma^2} \le e^{-a^2 U^2/2} e^{-a^2\gamma^2/2}$, with $U = T_0$ or $U = T$ respectively. This gives (18). For t sufficiently large (depending on a), choose $T = \sqrt{t}/a \ge T_0$. The second exponential in (18) is then 1, while

$$\eta_R(\sqrt{t}/a) \sim \frac{2^{2/3}}{57.54(\log t)^{2/3}(\log\log t)^{1/3}}.$$

Absorbing the bounded low-zero contribution and the constant tail, one obtains the claim for K_a. The relation (12) gives the claim for $\mathcal{T}_a[E]$. □

Remark 5.7 (An explicit, finite but enormous interval). Platt and Trudgian [11] verified the critical-line condition for all zeros with $|\gamma| \le T_0 = 3 \cdot 10^{12}$. At $a = 0.2$ this gives $a^2 T_0^2 = 3.6 \cdot 10^{23}$ and $a^2 T_0^2/4 = 9 \cdot 10^{22}$. In fact $Z_{0.2,1/4} < 9$ can be obtained from a published explicit zero-counting

bound. For $b = 0.01$, let $N(u)$ be the number of zeros with $0 < \gamma \le u$, counted with multiplicity. Using $N(14) = 0$ (the first zero has height above 14) and the estimate of Bellotti and Wong [13],

$$N(u) \le \frac{u}{2\pi}\log\frac{u}{2\pi e} + 0.10076\log u + 0.24460\log\log u + 8.08344 \le \frac{u^2}{8\pi} + u \quad (u \ge 14),$$

where the last step uses elementary inequalities valid for $u \ge 14$. Stieltjes integration by parts therefore gives

$$\begin{aligned} Z_{0.2,1/4} &= 4b\int_{14}^{\infty} ue^{-bu^2}N(u)\,du \\ &\le 4b\int_{14}^{\infty} ue^{-bu^2}\left(\frac{u^2}{8\pi} + u\right)du \\ &= \frac{(1+1.96)e^{-1.96}}{0.04\pi} + 28e^{-1.96} + 10\sqrt{\pi}\,\operatorname{erfc}(1.4) < 9. \end{aligned}$$

Thus (17) yields the fully explicit inequality

$$\boxed{|K_{0.2}(t) - K_{0.2}^{\le 3\cdot 10^{12}}(t)| < 10\exp(-9\cdot 10^{22}) \quad (0 \le t \le 3.6\cdot 10^{23}).} \tag{20}$$

This estimate uses a rigorous finite-height zero verification and unconditional zero counting, not numerical summation of the zeros up to T_0. It does *not* establish the global two-point positivity condition or the Riemann hypothesis. The smaller Korobov–Vinogradov bound (19) is a consequence of known zero-free regions, not a new such region.

Lemma 5.8 (A two-zero Diophantine certificate)**.** *Let γ_1, γ_2 denote the first two positive ordinates of zeros of ζ on the critical line, and put $\alpha = \gamma_2/\gamma_1$. The integer pairs*

$$\begin{aligned} (p_-, q_-) &= (23817266139817790182574 4,\ 160141697142335224793 1741), \\ (p_+, q_+) &= (45311560269459790675370 35,\ 304664276710205314158 8344) \end{aligned}$$

satisfy $p_+q_- - p_-q_+ = -1$ and

$$q_-\alpha - p_- < -2.49735\cdot 10^{-25} < 0, \qquad q_+\alpha - p_+ > 1.49333\cdot 10^{-25} > 0. \tag{21}$$

Thus, for every pair of integers p, q with $1 \le q < q_+$,

$$|q\alpha - p| > 2.49735\cdot 10^{-25}. \tag{22}$$

Proof. The determinant is an exact integer calculation. To certify the numerical inequalities in (21), it suffices to isolate the two ordinates in intervals of radius 10^{-58} about

$$\begin{aligned} c_1 &= 14.13472514173469379045725198356247027078425711569924317568556746, \\ c_2 &= 21.02203963877155499262847959389690277733434052490278175462952040. \end{aligned}$$

These are rational decimal centers. One reproducible interval check is recorded in the ancillary script `certify_gaussian_zeros.py`. It evaluates the Hardy function $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2} + it)$ at $c_j \pm 10^{-58}$, using the Euler transform of the Dirichlet eta series truncated at $N = 350$ and directed interval arithmetic. If

$$b_n(s) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s}, \qquad s = \tfrac{1}{2} + it,$$

then the omitted eta-series tail satisfies the *explicit* absolute error bound

$$\left|\sum_{n=N}^{\infty} 2^{-n-1} b_n(s)\right| \le 2^{-N}\sqrt{\cosh(\pi t)},$$

by the Mellin-integral representation of $b_n(s)$ and $|\Gamma(\frac{1}{2}+it)|^2 = \pi/\cosh(\pi t)$. Consequently the computed Hardy-Z value has an additional error at most

$$\frac{2^{-N}\sqrt{\cosh(\pi t)}}{\sqrt{2}-1}.$$

The four endpoint enclosures have opposite signs across each interval, with absolute values exceeding 10^{-59} and the respective truncation bounds below $2\cdot 10^{-90}$. Continuity gives a critical-line zero in each interval; the identifications as the first two ordinates are standard and also follow from rigorous low-zero enumeration. Interval division then certifies (21). For an independently reproducible Arb check, the same ordinates can be enclosed in the following exact rational intervals:

$$\frac{N_1}{10^{56}} < \gamma_1 < \frac{N_1+1}{10^{56}}, \qquad \frac{N_2}{10^{56}} < \gamma_2 < \frac{N_2+1}{10^{56}},$$

where

$$N_1 = 14134725141734693790457251983562470270784257115699243175 68,$$
$$N_2 = 21022039638771554992628479593896902777334340524902781754 62.$$

Once these rational brackets are validated by a rigorous zero isolator, all subsequent residual inequalities follow from exact rational arithmetic. The ancillary script `certify_gaussian_zeros_arb_exact.py` performs both stages using Arb (`python-flint`) and Python's `Fraction`.

Finally, since the two integer vectors form a unimodular basis, $(p,q) = m(p_-,q_-) + n(p_+,q_+)$ for integers m,n. For $1 \le q < q_+$, either $n=0$ and $m \ge 1$, or m and n have opposite signs. Since the two residuals have opposite signs, $|q\alpha - p| = |m|\,|q_-\alpha - p_-| + |n|\,|q_+\alpha - p_+|$, which proves (22). □

Theorem 5.9 (Two-point positivity over a finite continuous interval)**.** *Let $a = 0.2$ and $L = 3.6\cdot 10^{23}$. Using the two-ordinate interval certificate in Lemma 5.8, one has*

$$G_{0.2}(t) \succeq 0 \quad (|t| \le L), \qquad |K_{0.2}(t)| < K_{0.2}(0) \quad (0 < |t| \le L).$$

More precisely, with $t_1 = \pi/(2\gamma_1)$,

$$K_{0.2}(0) - |K_{0.2}(t)| > 0.0547\, t^2 \qquad (0 < |t| \le t_1) \tag{23}$$
$$K_{0.2}(0) - |K_{0.2}(t)| > 5.24\cdot 10^{-57} \qquad (t_1 \le |t| \le L) \tag{24}$$

Proof. By evenness it suffices to consider $t > 0$. Set $T_0 = 3\cdot 10^{12}$ and $A_j = 2e^{-0.04\gamma_j^2}$. The rigorous verification of Platt and Trudgian [11] places all zeros of height at most T_0 on the critical line. Consequently

$$K_{0.2}^{\le T_0}(0) \pm K_{0.2}^{\le T_0}(t) = \sum_{0<\gamma\le T_0} 2e^{-0.04\gamma^2}(1 \pm \cos(t\gamma)) \ge A_1(1\pm\cos(t\gamma_1)) + A_2(1\pm\cos(t\gamma_2)),$$

where multiplicities are included and all summands are nonnegative.

For $0 < t \le t_1$, we have $\cos(t\gamma_1) \ge 0$, so the plus sign is bounded below by $A_1 > 6.765 \cdot 10^{-4}$. Since $\sin y \ge (2\sqrt{2}/\pi)y$ on $0 \le y \le \pi/4$, the minus sign is bounded below by

$$A_1(1 - \cos(t\gamma_1)) \ge \frac{4A_1\gamma_1^2}{\pi^2} t^2 > 0.05478\, t^2.$$

The unverified part $R(t) = K_{0.2}(t) - K_{0.2}^{\le T_0}(t)$ is even. For $0 < t \le t_1$ and $|\gamma| > T_0$, $|\lambda_\rho| \le 1.01|\gamma|$, $t|\lambda_\rho| \le 0.01\gamma^2$, and $|1 - \cosh z| \le \frac{1}{2}|z|^2 e^{|z|}$. Together with $\gamma^2 e^{-0.01\gamma^2} \le 100/e$, $Z_{0.2,1/4} < 9$ from Remark 5.7, and the even zero sum, these give

$$|R(0) - R(t)| \le e^{0.01} \frac{1.01^2}{2} \frac{100}{e} Z_{0.2,1/4}\, e^{-9\cdot 10^{22}} t^2 < 171 e^{-9\cdot 10^{22}} t^2.$$

For the plus sign, the coarser bound $|R(0) + R(t)| < 20e^{-9\cdot 10^{22}}$ suffices. Since $A_1 > 0.0547 t_1^2$, both eigenvalues exceed $0.0547t^2$, proving (23).

Now let $t_1 \le t \le L$. Choose nearest integers q, p to $t\gamma_1/\pi, t\gamma_2/\pi$ respectively, and write $t\gamma_1 = q\pi + \delta_1$, $t\gamma_2 = p\pi + \delta_2$, with $|\delta_1|, |\delta_2| \le \pi/2$. The ordinate enclosures give

$$1 \le q \le \left\lfloor L\gamma_1/\pi + \tfrac{1}{2} \right\rfloor = 161972019039770454545515 5 < q_+.$$

For either sign and every integer m, $1 \pm \cos(m\pi + \delta) \ge 1 - \cos\delta \ge 2\delta^2/\pi^2$ on $|\delta| \le \pi/2$. As $\delta_2 - \alpha\delta_1 = (q\alpha - p)\pi$, Cauchy–Schwarz and Lemma 5.8 imply

$$\begin{aligned} K_{0.2}^{\le T_0}(0) \pm K_{0.2}^{\le T_0}(t) &\ge \frac{2}{\pi^2}(A_1\delta_1^2 + A_2\delta_2^2) \\ &\ge \frac{2|q\alpha - p|^2}{\alpha^2/A_1 + 1/A_2} > 5.24705 \cdot 10^{-57}. \end{aligned}$$

By (20), $|R(0) \pm R(t)| < 20e^{-9\cdot 10^{22}}$, far less than $7 \cdot 10^{-60}$; hence both eigenvalues are greater than $5.24 \cdot 10^{-57}$. This proves (24) and the theorem. □

Remark 5.10 (Scope and computational status of the finite-range theorem). The conclusion is unconditional once the two explicit low-zero ordinate enclosures are verified by outward-rounded interval arithmetic. Both the directed-interval Euler–eta script `certify_gaussian_zeros.py` and the Arb/exact-rational script `certify_gaussian_zeros_arb_exact.py` are provided in the ancillary files for independent verification. The theorem has a *finite* time range, even though the endpoint is very large, and does not establish RH.

Remark 5.11 (What has and has not been compressed). The theorem reduces the *test family* from a class of functions to translations of one fixed Gaussian and to 2×2 matrices. The continuum of separations t remains. Proving the uniform inequality in the corollary is equivalent to proving RH, so the main difficulty has not been removed. The formulation does not solve RH, and its role here is expository and analytical [7].

6 Weil's quadratic form and related criteria

6.1 Exactly two Gaussian translates

Let $\phi_a(x) = (a\sqrt{2\pi})^{-1} e^{-x^2/(2a^2)}$, so that $\widehat{\phi_a}(z) = e^{-a^2 z^2/2}$ under convention (1). For $t_1, t_2 \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{C}$, set

$$g(x) = c_1\phi_a(x - t_1) + c_2\phi_a(x - t_2), \qquad \widetilde{\overline{g}}(x) = \overline{g(-x)}.$$

Define the spectral explicit-formula functional, on these admissible Schwartz–Gaussian test functions, by

$$\mathcal{W}(F) = \sum_{\rho} \widehat{F}(z_\rho).$$

Absolute convergence permits expanding the product:

$$\begin{aligned}\mathcal{W}(g * \widetilde{\overline{g}}) &= \sum_{\rho} e^{-a^2 z_\rho^2} \sum_{i,j=1}^{2} c_i \overline{c_j} e^{i(t_i - t_j) z_\rho} \\ &= \sum_{i,j=1}^{2} c_i \overline{c_j} K_a(t_i - t_j) = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} K_a(0) & K_a(t_1 - t_2) \\ K_a(t_1 - t_2) & K_a(0) \end{pmatrix} \begin{pmatrix} \overline{c_1} \\ \overline{c_2} \end{pmatrix}.\end{aligned}$$

The middle equality uses the symmetry of the complete zero multiset. Thus positivity of this form on *all two-translate choices* (c_1, c_2, t_1, t_2), with a fixed once and for all, is precisely condition (ii) of Theorem 3.1. Importantly, a Gaussian is not compactly supported; the displayed identity is justified here directly through the absolutely convergent explicit formula and should not be silently identified with an assertion restricted to Weil's original compactly supported test space.

For a general even continuous K, two-point positivity does not imply positive definiteness of all matrices $[K(t_i - t_j)]_{i,j=1}^{N}$. The implication back to full positivity *for this particular* K_a comes instead from the meromorphic Laplace-pole argument and positivity of the on-line Gaussian weights. This is a criterion about the zeta zero multiset, not a new form of Bochner's theorem.

6.2 Relation to classical zero traces and Suzuki's screw function

Cramér's study of zeta-zero traces and Guinand's Fourier-transform approach provide early historical context [2, 3]. The passage between Gaussian-smoothed prime powers and zero sums in Section 4 is an explicit-formula calculation, not a new prime-counting estimate. The equivalence between a suitable bounded growth statement and the absence of off-critical zeros is an instance of a classical analytic mechanism: a pole in the open right half-plane is incompatible with a Laplace transform of polynomially bounded data.

Suzuki [7, Theorems 1.2, 1.6 and 1.7] constructs an *unregularized* screw function and proves RH equivalences through screw-function positivity, boundedness of the associated function $\Psi(t)$, and pointwise nonnegativity of $\Psi(t)$. In particular, reducing RH to a one-variable inequality is not unprecedented. In our formulation, the Gaussian multiplier is nonvanishing at every complex frequency, and the particular *stationary* two-translate matrix suffices because off-line zeros force unbounded exponential modes. These comparisons support the expository framing of the present note; they do not establish priority for this exact Gaussian packaging.

6.3 Not the de Bruijn–Newman heat flow

Gaussian regularization of a *zero measure* is distinct from the de Bruijn–Newman deformation of the Fourier integral defining the Riemann Ξ-function. In the latter theory RH is equivalent to $\Lambda_{\mathrm{dBN}} \leq 0$, whereas Rodgers–Tao established $\Lambda_{\mathrm{dBN}} \geq 0$ [8] and the published Polymath work established $\Lambda_{\mathrm{dBN}} \leq 0.22$ [9]. Neither bound proves RH. No identification of K_a with a trace of an existing self-adjoint Hilbert–Pólya operator is asserted.

7 Validated pointwise evaluation by ball arithmetic

This section records a reproducible *pointwise* interval-evaluation procedure for the unconditional arithmetic expression (8). It must not be confused with a proof of the inequality in Theorem 3.1 for every $t \in \mathbb{R}$. The implementation uses Arb's midpoint–radius ball arithmetic, including validated numerical integration [1]. Its principal arithmetic operations do not use binary floating-point approximations of input decimal parameters.

Write $K_a(t) = P_a(t) + H_a(t) - S_a(t)$, where P_a denotes the pole term, H_a the archimedean integral, and S_a the nonnegative prime-power sum of (8) (for real t). The finite parts of H_a and S_a are evaluated with outward-rounded ball arithmetic. We record explicit bounds for their omitted tails.

Lemma 7.1 (Archimedean tail bound)**.** *For $a > 0$, $U > 0$, and real t, set $\Omega(u) = \operatorname{Re}\psi(\frac{1}{4} + iu/2) - \log\pi$. Then*

$$\left|\frac{1}{\pi}\int_U^\infty e^{-a^2u^2}\cos(tu)\Omega(u)\,du\right| \le \frac{9}{a\sqrt{\pi}}\operatorname{erfc}(aU) + \frac{5}{\pi a^2}e^{-a^2U^2}. \tag{25}$$

Proof. The classical series identity $\psi(z) - \psi(1) = (z-1)\sum_{n\ge0}[(n+1)(n+z)]^{-1}$, valid for $\operatorname{Re} z > 0$, implies $|\Omega(u)| \le 18 + 10u$ for $u \ge 0$. Indeed, the series is bounded by $\sum_{n\ge0}(n+\frac{1}{4})^{-2} < 20$, $|\frac{1}{4} + iu/2 - 1| \le \frac{3}{4} + u/2$, and $|\psi(1)| + \log\pi < 3$. Using $|\cos(tu)| \le 1$ and integrating $18 + 10u$ against $e^{-a^2u^2}$ gives (25). □

Lemma 7.2 (Prime-power tail bound)**.** *For $a > 0$, $t \ge 0$, $N \ge 2$, and $L = \log N \ge \max\{2, t\}$, define*

$$\mathcal{F}_a(L,c) = 2a^2e^{-(L-c)^2/(4a^2)} + ca\sqrt{\pi}\,\operatorname{erfc}\left(\frac{L-c}{2a}\right).$$

If $R_N(a,t)$ is the $n > N$ contribution to the prime-power term in (8), then

$$0 \le R_N(a,t) \le \frac{e^{a^2/4}}{2a\sqrt{\pi}}\left[e^{t/2}\mathcal{F}_a(L, a^2+t) + e^{-t/2}\mathcal{F}_a(L, a^2-t)\right]. \tag{26}$$

Proof. Use $0 \le \Lambda(n) \le \log n$. The functions $(\log x)x^{-1/2}\exp[-(\log x \mp t)^2/(4a^2)]$ are decreasing for $x \ge N$ under the stated assumption; hence their sums over $n > N$ are bounded by their integrals from N to infinity. Under $u = \log x$, the resulting integrals are $\int_L^\infty ue^{u/2-(u\mp t)^2/(4a^2)}du$. Completing the square and integrating gives (26). □

For the parameters $a = 0.2$, $U = 60$, and $N = 2000$, (25) gives an unconditional bound 1.187×10^{-61} for the omitted archimedean integral. The prime-power bound (26) is at most 4.890×10^{-85} for $0 \le t \le 2$. The former bound is deliberately conservative; it follows from $|\Omega(u)| \le 18 + 10u$. Replacing it by a logarithmic digamma bound requires a separate proof and cannot be done by pulling $\log(U+2)$ outside an integral over $[U, \infty)$, since the logarithm increases.

Table 1 records numerical centers and the radii *reported* from an Arb execution of the ancillary script with 65 decimal-digit working precision, $a = 0.2$, $N = 2000$, and $U = 60$. The finite archimedean integral is partitioned into subintervals of length at most 3, with each integral evaluated using Arb's validated quadrature. For formal computational certification, readers should retain and archive the *full ball enclosures*, the versions of Python and `python-flint`, and the script's unaltered output. The printed decimal centers and rounded radii below are not substitutes for that raw audit record. In particular, rounding a center to 18 decimal places introduces up to 5×10^{-19} additional absolute uncertainty; one must not pair that rounded center directly with a 10^{-38} ball radius and call the result a certified interval.

t	Pole	Arch.	Prime	Displayed $K_{0.2}(t)$	Reported Arb radius
0	+2.0201003342	−1.9498276286	+0.0695960910	+0.000676614553947953	$2.6 \cdot 10^{-38}$
0.5	+2.0835579477	−1.4360947511	+0.6469842235	+0.000478973133559730	$5.0 \cdot 10^{-38}$
1	+2.2779175891	−0.7560652980	+1.5218506628	+0.000001628265735973	$4.1 \cdot 10^{-40}$
2	+3.1171777060	−0.3805188659	+2.7373354196	−0.000676579473549847	$1.1 \cdot 10^{-38}$

Table 1: Numerical summaries of Arb results as reported from an external execution. Component centers are rounded to ten decimal places; the K column shows decimal values rounded to 18 places. The final column reports *underlying ball* radii; these are *not* error radii around the displayed 18-place decimals. Decimal output rounding must be included separately when making an enclosure from this table. The positive prime-power contribution is subtracted in $K = P + H - S$.

Remark 7.3 (Near-tangencies and spectral notch filtering). Under RH, write $0 < \gamma_1 \le \gamma_2 \le \cdots$ for the positive zero ordinates, counted with multiplicity, and put $A_n = 2e^{-a^2\gamma_n^2}$. Then the eigenvalues of $G_a(t)$ are

$$\lambda_+(t) = K_a(0) + K_a(t) = \sum_{n\ge1} A_n(1+\cos(t\gamma_n)) = 4\sum_{n\ge1} e^{-a^2\gamma_n^2}\cos^2\left(\frac{t\gamma_n}{2}\right),$$
$$\lambda_-(t) = K_a(0) - K_a(t) = \sum_{n\ge1} A_n(1-\cos(t\gamma_n)) = 4\sum_{n\ge1} e^{-a^2\gamma_n^2}\sin^2\left(\frac{t\gamma_n}{2}\right).$$

For $a = 0.2$, the first two weights are $A_1 \approx 6.76572455418 \times 10^{-4}$ and $A_2 \approx 4.20713503400 \times 10^{-8}$; the first contributes about 99.9938% of $K_{0.2}(0) \approx 0.000676614553948$. At the nodes $t_k = k\pi/\gamma_1$, the γ_1 contribution to λ_+ vanishes for odd k, while its contribution to λ_- vanishes for even k. Thus these eigenvalues act as *spectral notch filters*.

For the ninth node $t_9 \approx 2.000345503630$, very close to the $t = 2$ entry in Table 1, this gives

$$\lambda_+(t_9) = \sum_{n\ge2} A_n(1+\cos(t_9\gamma_n)) \approx 2.7298073180 \times 10^{-8}.$$

Of this, the γ_2 term contributes approximately $2.724446271 \times 10^{-8}$, or 99.804% of the gap. Numerically evaluating the *unconditional prime-side expression* $K = P + H - S$ at this node gives

$$P_{0.2}(t_9) \approx 3.117587869565, \quad H_{0.2}(t_9) \approx -0.380446750244,$$
$$S_{0.2}(t_9) \approx 2.737817706576, \quad K_{0.2}(t_9) \approx -0.000676587256.$$

In particular, its agreement with the spectral approximation is a numerical cross-check, not a proof of RH.

The fourth node $t_4 = 4\pi/\gamma_1 \approx 0.889042446058$ has an even narrower gap:

$$\lambda_-(t_4) = \sum_{n\ge2} A_n(1-\cos(t_4\gamma_n)) \approx 5.9138692814 \times 10^{-10}.$$

Here $1 - \cos(t_4\gamma_2) \approx 0.01278393$, and the γ_2 and γ_3 terms account for approximately 90.945% and 9.055% of this gap, respectively. Direct evaluation of the prime-side terms yields

$$P_{0.2}(t_4) \approx 2.222993564830, \quad H_{0.2}(t_4) \approx -0.853659245759,$$
$$S_{0.2}(t_4) \approx 1.368657705109, \quad K_{0.2}(t_4) \approx +0.000676613963.$$

The nodes cancel the first-zero contributions exactly *under* RH, but need not be stationary points of the full kernel. These numerical observations do not certify either gap: such a claim requires full, validated prime-side enclosures (including rounding and truncation errors), and interval certification on an entire t-interval requires additional uniform bounds.

As a supplementary comparison, the ancillary script computes the partial critical-line zero trace

$$K_a^{(25)}(t) = 2\sum_{n=1}^{25} e^{-a^2\gamma_n^2}\cos(t\gamma_n).$$

This is *not* automatically an enclosure of the entire zero sum: a bound for unlisted zeros, including the possible off-critical ones, requires its own zero-counting and zero-verification argument. Moreover, a reported enclosure for $K_a(t)$ with radius of order 10^{-38} does not establish a difference from $K_a^{(25)}(t)$ smaller than 10^{-61}. Such a claim would require an additional independent enclosure at correspondingly higher precision.

Finite-time interval certification is valuable for checking the explicit formula, but it cannot prove the uniform inequality of Theorem 3.1. The ancillary script is a computational reproducibility aid, not a proof of the Riemann hypothesis.

8 Research questions and conclusion

The following questions are distinct from the equivalence theorem:

1. Can the smoothed prime-error integral $\int_1^\infty E(u)W'_{a,t}(u)\,du$ in Proposition 5.2 be bounded uniformly (or polynomially) in $t \geq 0$ by an arithmetic argument that does not invoke a hypothesis already equivalent to RH?

2. Can new arithmetic input produce an unconditional estimate beyond the known zero-free-region transfer bound in Proposition 5.6, rather than merely improving the earlier convolution-based baseline?

3. Does the Gaussian parameter a admit a rigorous adaptive error analysis for finite computations, balancing zero and prime-power truncations?

4. Can a more exhaustive comparison with existing Gaussian zero traces and screw-function criteria locate an earlier equivalent two-translate statement, or identify a useful computational distinction?

The conclusion is exact but deliberately limited: the spectral reality of the zeta zeros is equivalent to boundedness of a single Gaussian-regularized zero trace, and thus to positivity of its two-point matrices. The prime-side explicit formula rewrites that criterion arithmetically. The convolution and zero-splitting representations make explicit two distinct unconditional baselines, including the stronger bound obtained by transferring known zero-free regions through Gaussian weights. The significance of the presentation is conceptual and expository; it is not a proof of the Riemann hypothesis or evidence of a new unconditional prime-number estimate.

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